→ An extension of (G, μ) on $\hat{R}^X$ a grouping $(\mathcal{H}, \nu)$ on Y together w/ Borel map $f: Y \to X$ s.t. ① induces graph hom

② f restricted to an acomp of $\mathcal{H}$ it is a graph iso onto image

③ $f_* \nu = \mu$

G is a quasi-factor of $\mathcal{H}$ if ① ② ③ $f_* \nu = \mu$

Measured prop. (T) (partly j/w Ł. Grabowski & S. Mellick)

Def'n ("Connes-Weiss"): A u.l.f. graphing $(\mathcal{G},\mu)$ has measured property (T) iff each of its ergodic extensions is strongly ergodic

→ A graphing $(\mathcal{G},\mu)$ is a

1. a Borel graph $\mathcal{G}\subset X\times X$ w/ countable comps.
2. (X,μ) is a standard prob space $\mathrm{cg}([0,1],\lambda)$
3. $\forall \alpha\in \mathrm{Aut}(X)$ s.t. $\forall^* x$ $\alpha(x)\sim_{\mathrm{Rel}(\mathcal{G})} x \Rightarrow \alpha_*\mu=\mu$.

"being in the same conn comp"

→ u.f.f.: $\exists D\in\mathbb{N}$, $\forall^* x$ $\deg(x)\le D$.

→ Ergodic: $\forall A\subset X$ Borel, $\mu(\partial_{\mathcal{G}}A)=0 \Rightarrow \mu(A)=0,1$

$\partial_{\mathcal{G}}A \subseteq \{x\in A \mid \exists y\notin A \text{ w/ } x\sim_{\mathcal{G}} y\}$

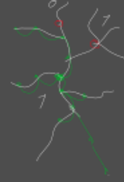
→ Strong ergodicity: $\forall (A_n)_n\subset X$ of Borel sets $\mu(\partial_{\mathcal{G}}A_n)\to 0$

$\Rightarrow \mu(A_n)(1-\mu(A_n))\to 0$

Thm (Hjorth-Kechris) A group Γ is non-amenable iff the Schreier graph for $\Gamma \curvearrowright (2^\Gamma, \{1/2, 1/2\}^{\text{prod}})$ is strongly ergodic.
↪ for a gen. $S \to E_G$ w/ edges $sx \sim x$.

Ex: Let (T, ρ) be a tree, then it admits an ergodic extension which is not strongly ergodic

SK: Let $(A_n)_n$ be a vanishing seq of nbhds, $A_n \subset X$, $\bigcap_n A_n = \emptyset$ and A_n is a complete section



WTS $\exists$ in some extension $(\mathcal{H}, \nu)$, $(R_n)_n \subset T$ s.t. $\nu(R_n) = 1/2$ but $\nu(\partial_{T_H} R_n) \to 0$

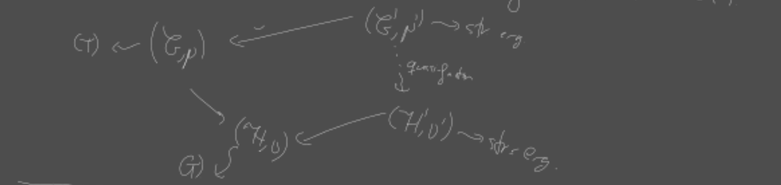
Take an extension by flipping a coin independently on each cluster. Then the R_n 's correspond to the set of x's whose cluster is coloured by 1

Thm (Pochat Connes-Weiss): Let G be an lcsc group.
Then G has (T) iff one (hence, all) of its ergodic pmp free actions have (T).

Example. Paln eq rel on a prop. (T) group

Prop: If $(\mathcal{G},\mu)$ has (T), then its quasi-factors have (T)

$$\begin{tikzcd}
(T) \arrow[r, phantom, "\curvearrowleft"] & (\mathcal{G},\mu) \arrow[dr] & & (\mathcal{G}',\mu') \arrow[ll] \arrow[d, dashed, "\text{quasi-factor}"] \arrow[r, phantom, "\rightsquigarrow \text{str erg}"] & \\
& & (\mathcal{H},\nu) \arrow[dl, squiggly] & (\mathcal{H}',\nu') \arrow[l] \arrow[r, phantom, "\rightsquigarrow \text{str erg.}"] & \\
& (T) & & &
\end{tikzcd}$$



Non-ex $(\mathcal{G},\mu)$ has planar comps $\Longrightarrow$ does not have (T)

① $\mathcal{G}$ needs to be bounded for (T)
② Thm

Q: What structures are compatible w/ Prop (T)?